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Approximation of Continuous and Discontinuous Functions by Generalized Sampling Series

P. L. BUTZER, S. RIES, AND R. L. STENS

Aachen University of Technology, Aachen, West Germany

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DEDICATED TO THE MEMORY OF GÉZA FREUD

1. INTRODUCTION

In his paper (*Acta Math. Acad. Sci. Hungar.* **5** (1954), 109–127) Über die Konvergenz des Hermite–Fejérschen Interpolationsverfahrens of 1954 Géza Freud considered the well-known Hermite–Fejér interpolation process $H_n f$ taken at the zeros of general orthogonal polynomials and gave conditions ensuring that $\lim_{n\to\infty} H_n f(t) = f(t)$. Here f is a bounded function defined on $[-1, 1]$ and continuous at t. Recently, I. V. Rybaltovskii [19] and independently R. Bojanic and F. W. Cheng [3] showed that the Hermite–Fejér process based on the zeros of the Chebyshev polynomials diverges as $n \to \infty$ provided f has a jump discontinuity at t. Moreover, there is a conjecture of Bojanic stating that any continuous interpolation process diverges at a jump discontinuity.

The aim of this paper is to consider the analogous questions for generalized sampling series given by

$$(S_W^\chi f)(t) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{W}\right) \chi(Wt-k) \qquad (t \in \mathbb{R};\ W>0), \tag{1.1}$$

where f is now a bounded function defined on $\mathbb{R}$. First, necessary and sufficient condition will be given upon χ such that

$$\lim_{W\to\infty} (S_W^\chi f)(t) = f(t) \tag{1.2}$$

at each point of continuity of f. It will turn out that (1.2) is essentially

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equivalent to $\sum_{k=-\infty}^{\infty} \chi(x-k)=1$, $x \in \mathbb{R}$, which can be rewritten via Poisson's summation formula in terms of the Fourier transform $\chi^{\wedge}$ as

$$\chi^{\wedge}(2k\pi)=\begin{cases} \dfrac{1}{\sqrt{2\pi}}, & k=0 \\ 0, & k \in \mathbb{Z}\setminus\{0\}. \end{cases}$$

Next the behaviour of (1.1) for $W \to \infty$ at a jump discontinuity will be investigated. In this respect,

$$\lim_{W \to \infty} (S_W^{\chi} f)(t)=\alpha f(t+0)+(1-\alpha) f(t-0) \tag{1.3}$$

if and only if $\sum_{k>x} \chi(x-k)=\alpha$. The latter condition can also be expressed equivalently in terms of the Fourier transform (cf. Theorem 2). The surprising fact here is that for continuous χ equality (1.3) implies $\chi(0)=0$, which in turn implicates that (1.1) cannot have the interpolation property

$$(S_W^{\chi} f)\left(\frac{k}{W}\right)=f\left(\frac{k}{W}\right) \qquad (k \in \mathbb{Z};\ W>0). \tag{1.4}$$

This confirms Bojanic's conjecture.

The paper also recalls some results from [12] on rates of convergence associated with (1.2), and gives some examples of functions χ. In particular, it is shown that if χ is discontinuous, then the convergence at jump discontinuities (1.3) is, however, compatible with the interpolation property (1.4).

2. Kernels for Generalized Sampling Series

Concerning notations, let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ be the sets of all naturals, all integers, all reals and all complex numbers, respectively. Let $B(\mathbb{R})$ denote the space of all bounded functions $f: \mathbb{R} \to \mathbb{C}$, and $C(\mathbb{R})$ the subspace of those $f \in B(\mathbb{R})$ which are uniformly continuous on $\mathbb{R}$, endowed which the usual supremum norm $\|\cdot\|$. Further, let $l^1(\mathbb{R})$ be those $\chi \in B(\mathbb{R})$ for which the series $\sum_{k=-\infty}^{\infty} |\chi(x-k)|$ converges uniformly on $[0, 1]$. Some properties of the latter class are listed in (cf. [12])

Lemma 1. *Let* $\chi \in l^1(\mathbb{R})$.

(a) $\sum_{k=-\infty}^{\infty} |\chi(x-k)|$ *converges uniformly on each compact subinterval of* $\mathbb{R}$.

(b) *Uniformly for all* $x \in \mathbb{R}$ *there holds*

$$\lim_{R \to \infty} \sum_{|x-k|>R} |\chi(x-k)|=0.$$

(c) *If χ is Lebesgue measurable, then it is also (absolutely) Lebesgue integrable over $\mathbb{R}$.*

(d) *For each $f \in B(\mathbb{R})$ the series (1.1) converges uniformly on compact intervals with respect to t and with respect to W.*

Our first result on the convergence behaviour of the series (1.1) for $W \to \infty$ is given in

THEOREM 1. (a) *The following two assertions are equivalent for $\chi \in L^1(\mathbb{R})$:*

(i) *for each $f \in B(\mathbb{R})$ and each point $t \in \mathbb{R}$ of continuity of f there holds*

$$\lim_{W \to \infty} (S_W^\chi f)(t) = f(t),$$

(ii) $\sum_{k=-\infty}^{\infty} \chi(x-k) = 1 \quad (x \in [0, 1))$.

(b) *If $\chi \in l^1(\mathbb{R})$ is continuous, then each of the assertions of part* (a) *is equivalent to*:

(iii)

$$\chi^\wedge(2k\pi) = \begin{cases} \dfrac{1}{\sqrt{2\pi}}, & k = 0 \\ 0, & k \in \mathbb{Z} \setminus \{0\}, \end{cases}$$

where $\chi^\wedge(v) := (1/\sqrt{2\pi}) \int_{-\infty}^{\infty} \chi(u)\, e^{-ivu}\, du$, $v \in \mathbb{R}$, is the Fourier transform of χ.

The proof of (ii) $\Rightarrow$ (i) follows by standard arguments using Lemma 1(b); the converse is immediate by choosing $f(x) \equiv 1$. For the equivalence of (ii) and (iii) see [12]; cf. also the proof of Theorem 2(ii*) $\Leftrightarrow$ (iv*) below.

A function $\chi \in l^1(\mathbb{R})$ satisfying (i) or equivalently (ii) of Theorem 1 is said to be a *kernel* (for a generalized sampling series). If it is continuous, it is called a *continuous* kernel. Hence a kernel has properties (i) and (ii) by definition, and a continuous kernel additionally property (iii).

Whereas Theorem 1 deals with the approximation of a function f by the series $S_W^\chi f$ at a point of continuity, the next aim is to investigate this series at a jump discontinuity of f, i.e., at a point t where the limits

$$f(t+0) := \lim_{\varepsilon \to 0+} f(t+\varepsilon), \qquad f(t-0) := \lim_{\varepsilon \to 0+} f(t-\varepsilon)$$

exist and are different. For a kernel χ the functions

$$\psi_\chi^+(x) := \sum_{k<x} \chi(x-k), \qquad \psi_\chi^-(x) := \sum_{k>x} \chi(x-k) \qquad (x \in \mathbb{R})$$

will be needed. These two series converge for all $x \in \mathbb{R}$ and represent functions with period one.

LEMMA 2. *Let χ be a continuous kernel. Then ψ_χ^- is continuous from the right at the integers, and ψ_χ^+ is continuous from the left there.*

The proof for χ_χ^- follows easily from the representation

$$\psi_\chi^-(x) = \sum_{k=-n+1}^{\infty} \chi(x-k) \qquad (x \in [n, n+1))$$

and the uniform convergence of the latter series on $[n, n+1)$, $n \in \mathbb{Z}$ (cf. Lemma 1a). The result for ψ_χ^+ can be proved similarly.

As a counterpart of Theorem 1 one now has

THEOREM 2. *Let $f \in B(\mathbb{R})$ have a jump at $t \neq 0$, and let $\alpha \in \mathbb{C}$.*

(a) *The following three assertions are equivalent for a kernel χ:*

(i) $\lim_{\substack{W \to \infty \\ Wt \notin \mathbb{Z}}} (S_W^\chi f)(t) = \alpha f(t+0) + (1-\alpha) f(t-0)$,

(ii) $\psi_\chi^-(x) = \alpha \qquad (x \in (0, 1))$,

(iii) $\psi_\chi^+(x) = 1 - \alpha \qquad (x \in (0, 1))$.

(b) *If χ is a continuous kernel, then the following five assertions are equivalent*:

(i*) $\lim_{W \to \infty} (S_W^\chi f)(t) = \alpha f(t+0) + (1-\alpha) f(t-0)$,

(ii*) $\psi_\chi^-(x) = \alpha \qquad (x \in [0, 1))$,

(iii*) $\psi_\chi^+(x) = 1 - \alpha \qquad (x \in [0, 1))$,

(iv*) $(1/\sqrt{2\pi}) \int_{-\infty}^0 \chi(u)\, e^{-i2k\pi u}\, du = \begin{cases} \alpha/\sqrt{2\pi}, & k = 0, \\ 0, & k \in \mathbb{Z} \setminus \{0\}, \end{cases}$

(v*) $(1/\sqrt{2\pi}) \int_0^\infty \chi(u)\, e^{-i2k\pi u}\, du = \begin{cases} (1-\alpha)/\sqrt{2\pi}, & k = 0, \\ 0, & k \in \mathbb{Z} \setminus \{0\}. \end{cases}$

Proof. Let us first set

$$g_t(x) := \begin{cases} f(x) - f(t-0), & x < t, \\ 0, & x = t, \\ f(x) - f(t+0), & x > t. \end{cases}$$

Then $g_t \in B(\mathbb{R})$ is continuous at zero, and hence by Theorem 1

$$\lim_{W \to \infty} (S_W^\chi g_t)(t) = 0. \tag{2.1}$$

Moreover, one has the representation

$$(S_W^\chi f)(t) = (S_W^\chi g_t)(t) + f(t-0) + \{f(t+0) - f(t-0)\}\, \psi_\chi^-(Wt) \quad (Wt \notin \mathbb{Z}). \tag{2.2}$$

Now, if (ii) is satisfied, then $\psi_\chi^-(x) = \alpha$ for all $x \in \mathbb{R}\setminus\mathbb{Z}$ since ψ_χ^- is periodic; (i) follows from (2.2) by letting $W \to \infty$, noting (2.1). Conversely, if (i) holds, then one has again by (2.2) that

$$\begin{aligned} &\alpha f(t+0) + (1-\alpha) f(t-0) \\ &\quad = f(t-0) + \{f(t+0) - f(t-0)\} \lim_{\substack{W \to \infty \\ Wt \notin \mathbb{Z}}} \psi_\chi^-(Wt). \end{aligned}$$

Since $f(t+0) \neq f(t-0)$, this implies

$$\lim_{\substack{W \to \infty \\ Wt \notin \mathbb{Z}}} \psi_\chi^-(Wt) = \alpha$$

or, equivalently for $x \in (0, 1)$, $n \in \mathbb{N}$,

$$\lim_{n \to \infty} \psi_\chi^-(x+n) = \alpha.$$

Noting once more that ψ_χ^- has period one, assertion (ii) is obvious. The equivalence of (i) and (iii) can be established by similar arguments. As to part (b), if χ is continuous, then the series (1.1) is continuous with respect to $W > 0$ by Lemma 1(d), and so one can drop the restriction $Wt \notin \mathbb{Z}$ in (i). Similarly, since ψ_χ^- is continuous from the right at the integers by Lemma 2, it must be equal to α everywhere, in particular at zero; likewise $\psi_\chi^+(x) = 1 - \alpha$, $x \in [0, 1)$.

Before verifying the equivalence of (iv*) and (v*) with the other assertions we show

COROLLARY 1. *If χ is a continuous kernel satisfying* (i*), *then* $\chi(0) = 0$.

The proof follows immediately from (ii*), (iii*) above as well as (ii) of Theorem 1, noting that

$$1 = \sum_{k=-\infty}^{\infty} \chi(-k) = \psi_\chi^-(0) + \chi(0) + \psi_\chi^+(0) = \alpha + \chi(0) + 1 - \alpha.$$

This result is somewhat surprising. It means in particular that a continuous kernel with property (i*) cannot have the familiar bell-shaped graph around the origin. A typical kernel having this property is a dipole.

Now the proof of Theorem 2(b) will be completed. Setting $\chi_0(x) := \chi(x)$ for $x<0$ and $:=0$ for $x \geqslant 0$, then $\psi_{\chi}^{-}(x) = \sum_{k=-\infty}^{\infty} \chi_0(x-k)$ is a continuous function on $[0, 1)$ with period one. By Poisson's summation formula (cf. [4, p. 201]) its Fourier expansion is given by

$$\psi_{\chi}^{-}(x) \sim \sqrt{2\pi} \sum_{k=-\infty}^{\infty} \chi_0^{\wedge}(2k\pi)\, e^{i2k\pi x}$$

$$= \sum_{k=-\infty}^{\infty} \left\{ \int_{-\infty}^{0} \chi(u)\, e^{-i2k\pi u}\, du \right\} e^{i2k\pi x}.$$

Therefore $\psi_{\chi}^{-}(x) = \alpha$, $x \in [0, 1)$, if and only if the Fourier series reduces to the term for $k=0$, and this term is equal to α. This is exactly (ii*)$\Leftrightarrow$(iv*). Finally the equivalence (iv*)$\Leftrightarrow$(v*) follows from Theorem 1(iii).

In the following, when refering to the assertion (i) or (i*) we speak of convergence (of a generalized sampling series) at a jump discontinuity.

Let us recall a result concerning the order of approximation in regard to (1.2) (see [12]). Here we need the modulus of continuity of $f \in C(\mathbb{R})$,

$$\omega(\delta; f) := \sup\{ \|f(\cdot + h) - f(\cdot)\|;\ |h| < \delta\} \qquad (\delta > 0),$$

as well as the absolute moment of $\chi \in B(\mathbb{R})$ of order $\gamma \geqslant 0$, defined by

$$m_{\gamma}(\chi) := \left\| \sum_{k=-\infty}^{\infty} |(\cdot - k)^{\gamma} \chi(\cdot - k)| \right\|.$$

THEOREM 3. *Let $\chi \in C(\mathbb{R})$ be a kernel.*

(a) *The series* (1.1) *defines a family of bounded linear operators from $C(\mathbb{R})$ into itself, satisfying*

$$\|S_W^{\chi}\|_{[C(\mathbb{R}), C(\mathbb{R})]} = m_0(\chi),$$

$$\lim_{W \to \infty} \|S_W^{\chi} f - f\| = 0 \qquad (f \in C(\mathbb{R})).$$

(b) *If $m_1(\chi) < \infty$, then*

$$\|S_W^{\chi} f - f\| \leqslant M_1\, \omega(W^{-1}; f) \qquad (f \in C(\mathbb{R});\ W > 0).$$

(c) *If $m_{r+1}(\chi) < \infty$ for some $r \in \mathbb{N}$, then the following three assertions are equivalent*:

(i) $\|S_W^{\chi} f - f\| \leqslant M_2 W^{-r} \omega(W^{-1}; f^{(r)}) \qquad (f \in C^{(r)}(\mathbb{R});\ W > 0)$,

(ii) $\sum_{k=-\infty}^{\infty} (x-k)^j \chi(x-k) = 0 \qquad (x \in [0, 1);\ j = 1, 2, ..., r)$,

(iii) $[\chi^{\wedge}]^{(j)}(2k\pi) = 0 \qquad (k \in \mathbb{Z};\ j = 1, 2, ..., r)$.

The constants M_1 *and* M_2 *depend on* χ *and* r *only, and* $C^{(r)}(\mathbb{R}) := \{f \in C(\mathbb{R});\ f^{(r)} \in C(\mathbb{R})\}$.

For results concerning the rate of convergence in the case where only the absolute moment of order $0 < \gamma < 1$ is finite see [12].

Conditions of type (ii), (iii) of Theorems 1 and 3 are not new. They were already used in connection with convergence and stability results for finite element approximation. See Fix and Strang [8], Aubin [1, pp. 12, 131], for example; also Dahmen and Micchelli [6].

Since the series (1.1) may be regarded as a discretized version of the convolution integral

$$(I_W^\chi f)(t) := W \int_{-\infty}^{\infty} f(u)\, \chi[W(t-u)]\, du, \tag{2.3}$$

it is of interest to compare the results of Theorems 1, 2, and 3 with corresponding ones for the integral (2.3). It is well known that

$$\lim_{W \to \infty} (I_W^\chi f)(t) = f(t) \tag{2.4}$$

holds for suitable f and t if and only if

$$\int_{-\infty}^{\infty} \chi(u)\, du = 1, \qquad \text{or} \qquad \int_{-\infty}^{\infty} \chi(x-u)\, du = 1 \quad (x \in \mathbb{R}). \tag{2.5}$$

This equivalence is the counterpart of (i)$\Leftrightarrow$(ii) of Theorem 1, the series in the latter being replaced by integrals. Rewriting (2.5) as $\chi^\wedge(0) = 1/\sqrt{2\pi}$, then (2.4)$\Leftrightarrow$(2.5) corresponds to (i)$\Leftrightarrow$(iii) of Theorem 1. However, in case of the integral (2.3) there is only one condition upon the Fourier transform $\chi^\wedge(v)$, namely at $v = 0$, whereas in case of the series there are countably many such conditions. Hence it follows that any χ satisfying (i) of Theorem 1 also satisfies (2.4), but not conversely.

Similarly one has (cf. [2, p. 23])

$$\lim_{W \to \infty} (I_W^\chi f)(t) = \alpha f(t+0) + (1-\alpha) f(t-0) \tag{2.6}$$

if and only if

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \chi(u)\, du \equiv \frac{1}{\sqrt{2\pi}} \int_{u>x} \chi(x-u)\, du = \frac{\alpha}{\sqrt{2\pi}} \qquad (x \in \mathbb{R}). \tag{2.7}$$

This equivalence can be compared with Theorem 2 in the same way. But here still another difference occurs between the series (1.1) and the integrals (2.3). Whereas (2.7) is always satisfied for some $\alpha \in \mathbb{C}$, condition (ii) of

Theorem 2 is not necessarily so. This means that (2.6) holds for every kernel χ whenever the right-hand side is meaningful, but (i) of Theorem 2 is valid only under additional assumptions upon χ.

For counterparts of Theorem 3 for integrals see [4, Sect. 3.4].

3. Examples of Kernels

Let $\varphi \in B(\mathbb{R})$ be continuous and absolutely integrable over $\mathbb{R}$ such that $\varphi^\wedge(0) = 1/\sqrt{2\pi}$ and $\varphi^\wedge(v) = 0$ for $|v| \geqslant 2\pi$. In this case φ is an entire function of exponential type $\leqslant 2\pi$, or, a so-called bandlimited function. Since $\varphi \in l^1(\mathbb{R})$ (cf. [10, p. 124]), one easily obtains by Theorem 1 that φ is a kernel. Furthermore, φ is uniformly continuous on $\mathbb{R}$, i.e., $\varphi \in C(\mathbb{R})$, and one has by Theorems 1 and 3

Corollary 2. *Let φ be given as above.*

(a) *If $f \in B(\mathbb{R})$ is continuous at $t \in \mathbb{R}$, then*

$$\lim_{W \to \infty} (S_W^\varphi f)(t) = f(t).$$

(b) *For $f \in C(\mathbb{R})$ there holds*

$$\lim_{W \to \infty} \|S_W^\varphi f - f\| = 0.$$

It should be mentioned that applications of Theorem 3(b) or (c) do not necessarily give best possible estimates. In this respect see [5, 14–16] where direct and inverse approximation theorems as well as saturation theorems are to be found.

Particular examples of bandlimited kernels are

$$\varphi_1(x) := \frac{1}{2\pi}\left(\frac{\sin x/2}{x/2}\right)^2 \qquad \text{(Fejér's kernel)},$$

$$\varphi_2(x) := \frac{3}{2\pi} \cdot \frac{\sin(x/2)\sin(3x/2)}{3x^2/4} \qquad \text{(de la Vallée Poussin's kernel)},$$

$$\varphi_3(x) := \frac{\sin(\pi x/2)\sin \pi x}{(\pi x)^2/2}.$$

Corollary 1 immediately yields that the sampling series associated with these kernels cannot converge at jump discontinuities. This is also a particular case of a result which will be proved below. In fact, it will be seen that no bandlimited kernel at all can have property (i) or (i*) of Theorem 2.

Next take $\chi = B_n$, $n = 2, 3,\ldots$, where B_n is the central B-spline of degree n given via its inverse Fourier transform by (cf. [13, p. 12])

$$B_n(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} B_n^{\wedge}(v)\, e^{ivx}\, dv, \qquad B_n^{\wedge}(v) := \frac{1}{\sqrt{2\pi}} \left(\frac{\sin v/2}{v/2}\right)^n. \quad (3.1)$$

Since $B_n^{\wedge}$ is an entire function of exponential type $n/2$, it follows by the Paley–Wiener theorem that B_n has support $[-n/2, n/2]$; hence it belongs to $l^1(\mathbb{R})$. So it follows by Theorem 1 and Theorem 3 that

COROLLARY 3. (a) *If $f \in B(\mathbb{R})$ is continuous at $t \in \mathbb{R}$, then*

$$\lim_{W \to \infty} \sum_{k=-\infty}^{\infty} f\left(\frac{k}{W}\right) B_n(Wt-k) = f(t). \quad (3.2)$$

(b) *For $r = 0$ or $r = 1$ there holds*

$$\left\| \sum_{k=-\infty}^{\infty} f\left(\frac{k}{W}\right) B_n(Wt-k) - f(t) \right\|$$
$$\leqslant M_r W^{-r} \omega(W^{-1}; f^{(r)}) \qquad (f \in C^{(r)}(\mathbb{R});\ W > 0).$$

On the other hand, since

$$B_n(0) = \frac{2}{\pi} \int_0^{\infty} \left(\frac{\sin v}{v}\right)^n dv$$
$$\geqslant \frac{2}{\pi} \sum_{k=0}^{\infty} \int_0^{\pi} \left(\frac{\sin v}{v+2k\pi}\right)^n + (-1)^n \left(\frac{\sin v}{v+(2k+1)\pi}\right)^n dv > 0,$$

it follows by Corollary 1 that the series in (3.2) cannot converge at jump discontinuities. To overcome this disadvantage we will make use of

LEMMA 3. *Let χ_1, χ_2 be two continuous kernels having support in $[-a, a]$ and $[-b, b]$, respectively, and $\alpha \in \mathbb{C}$. Then*

$$\chi_3(x) := \alpha \chi_1(x-a) + (1-\alpha)\, \chi_2(x+b)$$

is again a continuous kernel satisfying

$$\lim_{\substack{W \to \infty \\ Wt \notin \mathbb{Z}}} \sum_{k=-\infty}^{\infty} f\left(\frac{k}{W}\right) \chi_3(Wt-k) = \alpha f(t+0) + (1-\alpha)\, f(t-0) \quad (3.3)$$

for every $f \in B(\mathbb{R})$ and every $t \in \mathbb{R}$ for which $f(t+0)$ and $f(t-0)$ exist.

The proof follows readily by Theorems 1 and 2, noting that

$$\chi_3^\wedge(v) = \alpha\chi_1^\wedge(v)\, e^{-iva} + (1-\alpha)\, \chi_2^\wedge(v)\, e^{ivb} \qquad (v \in \mathbb{R}),$$

$$\frac{1}{\sqrt{2\pi}} \int_0^\infty \chi_3(u)\, e^{-ivu}\, du = \frac{\alpha}{\sqrt{2\pi}} \int_0^\infty \chi_1(u-a)\, e^{-ivu}\, du = \alpha e^{-iva} \chi_1^\wedge(v) \qquad (v \in \mathbb{R}).$$

Applying now Lemma 3 to $\chi_1 = \chi_2 = B_n$, $a = b = n/2$, yields

COROLLARY 4. *For $f \in B(\mathbb{R})$ and $\alpha \in \mathbb{C}$ there holds*

$$\lim_{\substack{W \to \infty \\ Wt \notin \mathbb{Z}}} \sum_{k=-\infty}^{\infty} f\left(\frac{k}{W}\right) \{\alpha B_n(Wt - k - n/2) + (1-\alpha)\, B_n(Wt - k + n/2)\}$$
$$= \alpha f(t+0) + (1-\alpha)\, f(t-0)$$

provided $f(t+0)$ and $f(t-0)$ exist.

For further kernels built up from the B-splines B_n the reader is referred to [7, 11].

We conclude this section with an example of a discontinuous kernel. Let

$$C_2(x) := \tfrac{1}{2}\{B_2(x+2) + B_2(x-2)\}$$
$$= \begin{cases} (|x|-1)/2, & 1 \leqslant |x| < 2 \\ (3-|x|)/2, & 2 \leqslant |x| < 3 \\ 0, & \text{elsewhere}; \end{cases}$$

then C_2 is a continuous kernel satisfying (i*) of Theorem 2 with $\alpha = \frac{1}{2}$ by Corollary 4. Next consider the step function

$$S(x) := \begin{cases} 1, & |x| < 1 \\ -1, & 1 < |x| < 2 \\ -1/2, & |x| = 2 \\ 0, & \text{elsewhere}. \end{cases}$$

One easily verifies that $\sum_{k=-\infty}^{\infty} S(x-k) = 0$, $x \in [0, 1)$, and $\sum_{k>x} S(x-k) = 0$, $x \in (0, 1)$. Hence it follows that

$$D_2(x) := C_2(x) + S(x)$$

is again a kernel satisfying (ii) of Theorem 2 with $\alpha = \frac{1}{2}$.

COROLLARY 5. *For $f \in B(\mathbb{R})$ there holds*

$$\lim_{\substack{W \to \infty \\ Wt \notin \mathbb{Z}}} \sum_{k=-\infty}^{\infty} f\left(\frac{k}{W}\right) D_2(Wt-k) = \frac{1}{2}\{f(t+0)+f(t-0)\}$$

whenever the right-hand side is meaningful.

Note that the series associated with D_2 converges at jump discontinuities although $D_2(0)=1$. This is no contradiction to Corollary 1 since D_2 is discontinuous. We will return to this example in Section 5.

4. PREDICTION BY MEANS OF DURATION-LIMITED KERNELS

The theory of generalized sampling series established so far can also be used to compute function values at a point t by using only samples at points k/W with $k/W \leqslant t$ or even $k/W < t$. If the variable of t is regarded as time, this means that f can be computed at future time t if it is known in the past.

To carry this out one just needs to take a kernel χ with support contained in the right half-axis. More precisely, if the support of χ is contained in $[T_1, T_2]$ for some $0 \leqslant T_1 < T_2 \leqslant \infty$, then the sampling series takes the form

$$(S_W^{\chi} f)(t) = \sum_{k \in \mathbb{A}} f\left(\frac{k}{W}\right) \chi(Wt-k), \tag{4.1}$$

where the set $\mathbb{A} \subset \mathbb{Z}$ is given by

$$\mathbb{A} := \left\{k \in \mathbb{Z};\, t - \frac{T_2}{W} \leqslant \frac{k}{W} \leqslant t - \frac{T_1}{W}\right\}.$$

To construct those kernels we start with an arbitrary kernel having support in $[-a, a]$. Then for each $T \geqslant 0$

$$\chi^*(x) := \chi(x-T-a)$$

has support contained in $[T, T+2a]$; it is again a kernel. The latter follows from

$$[\chi^*]^{\wedge}(v) = e^{-iv(T+a)} \chi^{\wedge}(v) \qquad (v \in \mathbb{R})$$

and Theorem 1.

As an example, take $\chi(x) := B_2(x)$, and set for some $T \geqslant 0$

$$B_2^*(x) := B_2(x-1-T) = \begin{cases} x-T, & T \leqslant x < T+1 \\ 2+T-x, & T+1 \leqslant x < T+2 \\ 0, & \text{elsewhere} \end{cases}$$

COROLLARY 6. (a) *Let $f \in B(\mathbb{R})$, and t be a point of continuity of f. Then for arbitrary $T \geqslant 0$ there holds*

$$\lim_{W \to \infty} \sum_{k \in \mathbb{A}^*} f\left(\frac{k}{W}\right) B_2(Wt-k-1-T) = f(t), \tag{4.2}$$

where $\mathbb{A}^* := \{k \in \mathbb{Z}; t-(T+2)/W \leqslant k/W \leqslant t-T/W\}$.

(b) *If $f \in C(\mathbb{R})$, then (4.2) holds uniformly with respect to $t \in \mathbb{R}$.*

(c) *If $f \in C(\mathbb{R})$, then*

$$\left\| \sum_{k \in \mathbb{A}^*} f\left(\frac{k}{W}\right) B_2(Wt-k-1-T) - f(t) \right\| \leqslant M_1 \omega(W^{-1}; f) \qquad (W > 0).$$

Note that one needs the samples only for $k/W \leqslant t - T/W$, $T \geqslant 0$ arbitrary, and one finds f at the time t. But it should be pointed out that the constant M_1 depends on B_2^*, so in particular on T.

The convergence at jump discontinuities could also be handled in this frame.

5. SAMPLING SERIES DIVERGING AT JUMP DISCONTINUITIES

In this section it is shown that two important classes of continuous kernels cannot have the property (i*) of Theorem 2.

THEOREM 4. *If χ is a continuous kernel having the interpolation property (1.4) for each bounded function f, then the associated sampling series diverges for $W \to \infty$ at the jump discontinuities.*

The proof follows from the fact that the interpolation property is equivalent to

$$\chi(k) = \begin{cases} 1, & k=0 \\ 0, & k \in \mathbb{Z} \setminus \{0\}, \end{cases}$$

together with Corollary 1.

Theorem 4 can also be stated as follows. An interpolation process of the form (1.1) cannot approximate a function f at a jump discontinuity. A corresponding result for the Hermite–Féjer interpolation process was recently established by Bojanic and Cheng [3].

THEOREM 5. *If φ is a bandlimited kernel as described at the beginning of Section 3, then the limit relation (i*) of Theorem 2 cannot hold.*

Proof. Assume that (i*) is valid in this instance. Then by Theorem 2 one has that

$$\psi_\varphi^-(x) = \sum_{k=1}^{\infty} \varphi(x-k) = \alpha \qquad (x \in [0, 1)),$$

the series converging absolutely and uniformly on compact intervals. This implies by [10, p. 127] that $\sum_{k=1}^{\infty} \varphi(x-k)$ has an analytic extension to the whole complex plane, which must be equal to α; in particular

$$\sum_{k=1}^{\infty} \varphi(x-k) = \alpha \qquad (x \in \mathbb{R}).$$

Furthermore, for arbitrary $x \in \mathbb{R}$

$$\alpha = \int_x^{x+1} \sum_{k=1}^{\infty} \varphi(u-k)\, du = \sum_{k=1}^{\infty} \int_x^{x+1} \varphi(u-k)\, du = \int_{-\infty}^{x} \varphi(u)\, du, \quad (4.1)$$

the interchange of summation and integration being justified in view of the uniform convergence of the series involved. The desired contradiction now follows by considering (4.1), letting $x \to \infty$ and $x \to -\infty$, respectively. Indeed, $x \to \infty$ yields $\alpha = 1$, whereas $x \to -\infty$ yields $\alpha = 0$.

An analogous result for the classical Shannon sampling series was already proved by de la Vallée Poussin [18], and the particular case of Theorem 5 for Fejér's kernel φ_2 is due to Theis [19].

Let us mention that the continuity of χ in Theorem 4 is essential. For example, the (discontinuities) kernel D_2 of Section 3 produces a sampling series which not only converges at jump discontinuities but also has the interpolation property. Indeed,

$$D_2(k) = C_2(k) + S(k) = \begin{cases} 1, & k = 0 \\ 0, & k \in \mathbb{Z} \backslash \{0\}. \end{cases}$$

But this is not surprising. Take, e.g., the convolution integral with Fejér's kernel φ_2 of Section 3, denote it by $\sigma_W f$, and define

$$(\sigma_W^* f)(t) := \begin{cases} (\sigma_W f)(t), & t \neq k/W \\ f(t), & t = k/W \end{cases} \qquad (t \in \mathbb{R};\ W > 0;\ k \in \mathbb{Z}).$$

Then $(\sigma_W^* f)(t)$ is a family of discontinuities functions converging for $W \to \infty$ to $f(t)$ at a point of continuity and for $W \to \infty$, $Wt \notin \mathbb{Z}$, to $\frac{1}{2}\{f(t+0)+f(t-0)\}$ at a jump discontinuity. Nevertheless, there holds the interpolation property $(\sigma_W^* f)(k/W) = f(k/W)$, $k \in \mathbb{Z}$, $W > 0$. However, there is an essential difference between the $\sigma_W^* f$ and the series with kernel D_2. Indeed, the discontinuities of $\sigma_W^* f$ are removable, whereas those of the latter are not.

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